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On the resonant interaction of neutral disturbances in two inviscid shear flows

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The second-order resonant interaction of two disturbances which are neutrally stable on a linear basis is investigated for cases when the mean flow is, first, an inviscid, homogeneous jet and, secondly, a stably stratified, antisymmetric shear layer for which the linear eigen-solutions are regular. For the former case, the periodic nature of the neutral disturbances is unaffected by the interaction. For the latter, the interaction can lead to an $O(\epsilon^{\frac{1}{2}})$ temporal growth rate of one disturbance, where ϵ is a characteristic disturbance amplitude.

1. Introduction

Since the publication of the seminal paper by Phillips (1960), knowledge concerning the possible exchange of energy between resonantly interacting waves in fluids has developed rapidly for, at least, the case where no basic flow exists. Most of the results are discussed in the recent book by Phillips (1966). If the interaction between a finite number of discrete waves is considered, the growth of one wave takes place simultaneously with the diminution of another, and a conservation condition which relates the amplitudes of the interacting waves can usually be found.

When a mean flow exists, the consequences of resonant wave interaction might be even more interesting. The resonant interaction will then not only permit the exchange of energy between waves but will also affect the rate at which energy can be transferred from the mean flow to each disturbance, due to alteration of the Reynolds stress.

Raetz (1959; see also the discussion by Stuart 1962) has demonstrated that certain unstable, three-dimensional disturbances in Blasius flow fulfil the conditions for second-order resonance and that they would undergo consequently secular variation. However, Benney & Niell (1962) have argued that such variation should simply be the manifestation of energy exchange between the interacting disturbances.

In an attempt to clarify the matter, we shall consider two particular inviscid shear flows for which two disturbances, which are neutrally stable on a linear basis, interact resonantly when terms of the second order are considered. Self-interaction effects become important only when terms of the third order are considered. On the basis of a second-order analysis, we shall try to decide whether both of the waves might become unstable due to the resonant interaction, i.e.

whether they would exhibit unbounded growth to the order considered. If so, we shall conclude that energy can be transferred from the mean flow to the disturbances due to the interaction. On the other hand, if bounded behaviour results, we shall conclude that the disturbances take part in an energy-sharing process involving only themselves.

2. Analysis of wave interaction

We make all quantities non-dimensional on the basis of a length L, a velocity $\bar{U}_0$, and a density $\bar{\rho}_0$. In the following analysis, the parameter g should therefore be taken as

$$g = g_0 L/\bar{U}_0^2, \tag{2.1}$$

where g_0 is the gravitational constant.

It is considerably simpler to consider only two-dimensional flow. We shall show that resonant interaction can take place even with this restriction. If we define a stream function by

$$u = \partial\psi/\partial y, \quad v = -\partial\psi/\partial x, \tag{2.2}$$

then the equations which govern the motion of an inviscid, incompressible stratified flow are

$$D\rho/Dt = 0 \tag{2.3}$$

and

$$\rho\frac{D}{Dt}\nabla^2\psi - g\frac{\partial\rho}{\partial x} + \frac{\partial\rho}{\partial x}\frac{D}{Dt}\frac{\partial\psi}{\partial x} + \frac{\partial\rho}{\partial y}\frac{D}{Dt}\frac{\partial\psi}{\partial y} = 0, \tag{2.4}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}. \tag{2.5}$$

We now consider a parallel flow in the x-direction to be perturbed by a small disturbance of $O(\epsilon)$ and seek a solution by expanding in powers of ϵ, i.e., let

$$\rho = \rho_0(y) + \sum_{n=1} \epsilon^n \rho_n(x,y,t) \tag{2.6}$$

and

$$\psi = \psi_0(y) + \sum_{n=1} \epsilon^n \psi_n(x,y,t). \tag{2.7}$$

We shall also assume that

$$\left(\frac{d\rho_0}{dy}\right)\Big/\rho_0 \sim O(\beta) \quad (\beta \ll 1 \text{ but } g\beta \sim O(1)). \tag{2.8}$$

For the linear problem, the Boussinesq approximation is then made, and the last two terms in (2.4) are ignored. In order to use the approximation in the $O(\epsilon^2)$ analysis, we must then have

$$\epsilon \gg \beta. \tag{2.9}$$

We shall now restrict the investigation considerably by assuming that: (i) only neutrally stable disturbances are involved, (ii) the corresponding eigensolutions are regular throughout the flow field, and (iii) only two waves interact. Assumptions (i) and (iii) are made on the grounds of expediency; we might expect rather similar results to hold for the case of several resonantly interacting, unstable disturbances, although the analysis would become considerably more complicated. Assumption (ii), however, is more restrictive, and we shall discuss its implications later.

The linear solution can then be expressed as

$$\psi_1 = \phi_{1\mathrm{I}}(y)(A_\mathrm{I}E_\mathrm{I}+\tilde{A}_\mathrm{I}\tilde{E}_\mathrm{I})+\phi_{1\mathrm{II}}(y)(A_\mathrm{II}E_\mathrm{II}+\tilde{A}_\mathrm{II}\tilde{E}_\mathrm{II}) \tag{2.10}$$

and

$$\rho_1 = \rho_{1\mathrm{I}}(y)(A_\mathrm{I}E_\mathrm{I}+\tilde{A}_\mathrm{I}\tilde{E}_\mathrm{I})+\rho_{1\mathrm{II}}(y)(A_\mathrm{II}E_\mathrm{II}+\tilde{A}_\mathrm{II}\tilde{E}_\mathrm{II}), \tag{2.11}$$

where tilde denotes the complex conjugate and

$$E_j = \exp\{i\alpha_j(x-c_jt)\}, \quad \mathrm{Im}\,(c_j) = 0. \tag{2.12}$$

The functions $\rho_{ij}(y)$ and $\phi_{1j}(y)$ satisfy

$$\rho_{1j} = \rho_0'\phi_{1j}/(\psi_0'-c_j), \tag{2.13}$$

where a prime denotes d/dy, and

$$(\psi_0'-c_j)(\phi_{1j}''-\alpha_j^2\phi_{1j})-\psi_0'''\phi_{1j}-\left(\frac{g\rho_0'}{\rho_0}\right)\left(\frac{\phi_{1j}}{\psi_0'-c_j}\right) = 0, \tag{2.14}$$

where

$$\phi_{1j}\to 0 \quad \text{as} \quad y\to\pm\infty. \tag{2.15}$$

For future reference, (2.14) is expressed as

$$(\psi_0'-c_j)\,L_j\phi_{1j} = 0. \tag{2.16}$$

After dropping a term of $O(\epsilon\beta^2)$ in accordance with (2.9), the second-order problem is given by

$$\rho_0\left\{\frac{\partial}{\partial t}+\psi_0'\frac{\partial}{\partial x}\right\}\nabla^2\psi_2-\rho_0\psi_0'''\frac{\partial\psi_2}{\partial x}-g\frac{\partial\rho_2}{\partial x} = -\rho_0\left\{\frac{\partial\psi_1}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi_1}{\partial x}\frac{\partial}{\partial y}\right\}\nabla^2\psi_1 \tag{2.17}$$

and

$$\frac{\partial\rho_2}{\partial t}+\psi_0'\frac{\partial\rho_2}{\partial x}-\rho_0'\frac{\partial\psi_2}{\partial x} = -\frac{\partial\psi_1}{\partial y}\frac{\partial\rho_1}{\partial x}+\frac{\partial\psi_1}{\partial x}\frac{\partial\rho_1}{\partial y}. \tag{2.18}$$

The solution of this problem presents, in principle, no difficulties unless non-homogeneous terms arise which contain the linear eigenvalues in the form (2.12); this is the case of resonant interaction. Under assumption (iii), resonance can occur only if one wave-number is twice the other, say,

$$\alpha_\mathrm{II} = 2\alpha_\mathrm{I}, \quad c_\mathrm{I} = c_\mathrm{II} = c, \tag{2.19}$$

so that $E_\mathrm{II} = E_\mathrm{I}^2$ and $E_\mathrm{I} = E_\mathrm{II}\tilde{E}_\mathrm{I}$. These relations are naturally satisfied only for a few cases. Besides the case of Helmholtz instability (with c now complex), which is somewhat exceptional because (2.19) can be satisfied for all α_I, the following two cases result from a perusal of the survey article by Drazin & Howard (1966):

(A) a homogeneous jet,

$$\psi_0' = \mathrm{sech}^2 y, \quad \rho_0 = \text{constant}, \tag{2.20}$$

$$\alpha_\mathrm{I} = 1, \quad c_\mathrm{I} = \tfrac{2}{3}, \quad \phi_{1\mathrm{I}} = \mathrm{sech}\, y\tanh y, \tag{2.21}$$

$$\alpha_\mathrm{II} = 2, \quad c_\mathrm{II} = \tfrac{2}{3}, \quad \phi_{1\mathrm{II}} = \mathrm{sech}^2 y, \tag{2.22}$$

(for a jet, second-order resonant interaction is peculiar to the neutral disturbances, at least in two dimensions);

(B) a stratified, antisymmetric shear layer,

$$\psi_0' = U_0+\tanh y, \quad \rho_0 = \exp\{-\beta\tanh^3 y\}. \tag{2.23}$$

Miles (1963, figure 3) has shown that an infinite number of independent modes can exist for this case. Consider the first two modes which were found originally by Garcia (see Drazin & Howard 1966, pp. 77–8),

$$c_{\mathrm{I}} = U_0, \quad \phi_{1\mathrm{I}} = \tanh y\,(\operatorname{sech} y)^{\alpha_{\mathrm{I}}}, \quad J_0 = \alpha_{\mathrm{I}}(\alpha_{\mathrm{I}}+3)/3, \tag{2.24}$$

$$c_{\mathrm{II}} = U_0, \quad \phi_{1\mathrm{II}} = (\operatorname{sech} y)^{\alpha_{\mathrm{II}}}, \quad J_0 = (\alpha_{\mathrm{II}}-1)(\alpha_{\mathrm{II}}+2)/3, \tag{2.25}$$

where J_0 is an over-all Richardson number, equal to βg in the present notation. The conditions (2.19) must be met for a fixed value of J_0; this occurs for

$$J_0 = \tfrac{4}{3}, \quad \alpha_{\mathrm{I}} = 1, \quad \alpha_{\mathrm{II}} = 2. \tag{2.26}$$

For $J_0 > \frac{4}{3}$, other resonant cases, which involve more than the first two modes, can arise; some remarks concerning this possibility will be made later.

On the basis of known solutions to the linear stability problem for homogeneous flow, no other cases appear to be possible, even if three neutral, but two-dimensional, disturbances are considered. For stratified flow, other cases are possible, at least if the basic flow is antisymmetric. For instance, resonant interaction appears to be possible for Hølmboe's shear flow, which has been studied by Miles (1963). This flow is more interesting than that given in (2.23) because the local Richardson number is zero at $y = 0$ for (2.23). However, the neutral eigensolutions are singular for Hølmboe's case, and a proper analysis would seem to require inclusion of the diffusive effects allowed by viscosity and heat conduction. A much simpler analysis appears to be possible for the above cases.

The fact that the conditions (2.19) are satisfied does not necessarily rule out the existence of a periodic solution for ψ_2 which involves E_{I} and E_{II} (cf. the remarks by Longuet-Higgins (1963) concerning the interaction of two-dimensional surface waves). Let us therefore consider this possibility and seek a solution of the form

$$\psi_2 = \phi_{2\mathrm{I}}E_{\mathrm{I}} + \phi_{2\mathrm{II}}E_{\mathrm{II}} + \phi_{2\mathrm{III}}E_{\mathrm{I}}E_{\mathrm{II}} + \phi_{2\mathrm{IV}}E_{\mathrm{II}} + \text{conjugates}, \tag{2.27}$$

$$\rho_2 = \rho_{2\mathrm{I}}E_{\mathrm{I}} + \rho_{2\mathrm{II}}E_{\mathrm{II}} + \rho_{2\mathrm{III}}E_{\mathrm{I}}E_{\mathrm{II}} + \rho_{2\mathrm{IV}}E_{\mathrm{II}}^2 + \text{conjugates}. \tag{2.28}$$

The possibility of resonance arises only because E_{I} and E_{II} occur in (2.27, 2.28), and so we shall assume $\phi_{2\mathrm{III}}$ and $\phi_{2\mathrm{IV}}$ can be obtained. By use of (2.13)–(2.16), the following equations for $\phi_{2\mathrm{I}}$ and $\phi_{2\mathrm{II}}$ are found from (2.17), (2.18) as

$$(\psi_0' - c)L_{\mathrm{I}}\phi_{2\mathrm{I}} = A_{\mathrm{II}}\tilde{A}_{\mathrm{I}}F(y)\phi_{1\mathrm{I}}\phi_{1\mathrm{II}} \tag{2.29}$$

and

$$(\psi_0' - c)L_{\mathrm{II}}\phi_{2\mathrm{II}} = \tfrac{1}{2}A_{\mathrm{I}}^2F(y)\phi_{1\mathrm{I}}^2, \tag{2.30}$$

where

$$F(y) = \left(\frac{\psi_0'''}{\psi_0' - c}\right)' + 2\left\{\frac{g\rho_0'}{\rho_0(\psi_0' - c)^2}\right\}' + \frac{g\rho_0'\psi_0''}{\rho_0(\psi_0' - c)^3}. \tag{2.31}$$

We note that $F(y)$ contains no terms of $O(g\beta^2)$, as required by (2.9). We also note that the non-homogeneous terms are regular throughout the flow field for our two cases.

The homogeneous parts of (2.29) and (2.30) are satisfied by $\phi_{1\mathrm{I}}$ and $\phi_{1\mathrm{II}}$, respectively. A necessary and sufficient condition for a solution to exist to either equation is that the non-homogeneous terms be orthogonal to the solution,

Φ_{1j}, say, of the homogeneous adjoint equation (cf. Ince 1956, section 9.34), which, for the present case (cf. Kelly 1967, equations 4.18–4.19), is simply

$$\Phi_{1j} = \frac{\phi_{1j}}{\psi_0' - c}. \tag{2.32}$$

The orthogonality condition is then

$$M = \int_{-\infty}^{\infty} (\psi_0' - c)^{-1} F(y) \phi_{1\mathrm{II}} \phi_{1\mathrm{I}}^2 dy = 0. \tag{2.33}$$

It can be shown directly, at least for the homogeneous case, that the integrand is proportional to the vertical gradient of the Reynolds stress produced by the interaction, averaged over the basic wavelength. The boundary conditions require that the Reynolds stress vanish as $y \to \pm\infty$, and so (2.33) is simply a statement of this requirement (as is the Rayleigh stability integral with regard to linearly unstable disturbances). If (2.33) is not satisfied, we must allow the non-linear terms to exert a secular influence upon the solution so that the condition on Reynolds stress is satisfied.

One further point should be discussed. For case A and for mode II in case B, the inviscid adjoint functions are singular at the critical layers, and we should therefore allow for diffusive corrections in those regions for each function. In a problem of non-linear self-interaction for which a similar difficulty arises, Schade (1964) has shown that the result of including such a correction is equiva lent to interpreting in the correct manner any integral rendered singular through multiplication by the inviscid adjoint function. The correct interpretation is provided by use of the path prescribed by Lin (1955, section 8.5). Although Schade's analysis pertains to the hyperbolic-tangent flow profile, we shall assume that such an interpretation is also correct for the case of a jet. For our particular stratified flow, the interaction integral is not singular, and no such difficulty arises for that integral. For the more general stratified case, however, the non-homogeneous terms in (2.29) and (2.30) would be singular by themselves, and the whole analysis would have to be reconsidered.

Using the above argument for the case of the jet, we find

$$M = 12 \int_{-\infty}^{\infty} (\mathrm{sech}^2 y - \tfrac{2}{3})^{-1} \tanh^3 y \,\mathrm{sech}^6 y \, dy = 0. \tag{2.34}$$

The integrand is odd, and the residues at the critical layers are found to cancel if the path is taken in Lin's manner. We therefore conclude that a second-order, periodic solution is possible for case A.

For case B, with $\beta g = \frac{4}{3}$, we find

$$M = -\frac{4}{7} \int_{-\infty}^{\infty} \mathrm{sech}^6 y \, dy = -\frac{64}{105}. \tag{2.35}$$

For the antisymmetric shear layer considered, a periodic solution appears to be impossible.

The results make an interesting contrast because $\phi_{1\mathrm{I}}$ and $\phi_{1\mathrm{II}}$ are the same for the two cases, whereas the function $\{\psi_0'''/(\psi_0' - c)\}'$ differs only by a constant factor. Noting that the density terms in (2.31) behave like the first term for case B, the difference between the two cases appears to be due to the difference in the mean flow profile. A physical explanation of the origin of subharmonic oscillations in an antisymmetric shear layer, which is based upon the vorticity distribution of the mean flow and which points to the results obtained thus far, has been advanced by Browand (1965). In general, however, (2.33) indicates that the distribution of vertical velocity associated with each disturbance must also be considered.

In order to establish the consequences of resonant wave interaction for the antisymmetric flow, we shall consider separately the cases when the interaction exerts temporal and spatial influence. This is desirable due to a novel feature of the temporal case which occurs for our particular flow.

3. Interaction of waves with fixed spatial periodicity

In early analyses of resonant wave interactions, a second-order solution was usually sought which exhibited a linear dependence upon time, in addition to the basic periodic behaviour. It can be shown that such a solution is inadequate for the present problem but that a solution of the form

$$\psi_2 = (at^2\phi_{1\mathrm{I}} + bt\hat{\phi}_{2\mathrm{I}} + \phi_{2\mathrm{I}})E_{\mathrm{I}} + (dt\phi_{1\mathrm{II}} + \phi_{2\mathrm{II}})E_{\mathrm{II}} + \phi_{2\mathrm{III}}E_{\mathrm{I}}E_{\mathrm{II}} + \phi_{2\mathrm{IV}}E_{\mathrm{II}}^2 + \text{conjugates} \qquad (3.1)$$

is sufficient, although b remains arbitrary in the second-order analysis. The reason for the quadratic dependence upon time will be given shortly. The importance of (3.1) lies in its interpretation, which is that the secular variation, at least with regard to mode I, takes place over the scale $\epsilon^{-\frac{1}{2}}$ rather than over the customary scale ϵ^{-1}. Using this fact, we shall investigate the problem by allowing A_{I} and A_{II} in (2.10) to become functions of the variable

$$\tau = \epsilon^{\frac{1}{2}}t.$$

We must then redefine (2.7) so that

$$\psi = \psi_0(y) + \sum_{n=1} \epsilon^{\frac{1}{2}(n+1)}\psi_{\frac{1}{2}(n+1)}(x, y, t, \tau), \qquad (3.2)$$

and use a similar expansion for the density in place of (2.6). This expansion is applicable as long as we are interested only in the initial stages of any instability.

The order $\epsilon^{\frac{3}{2}}$ solution for the stream function may be taken in the form (a similar expression is obtained for $\rho^{\frac{3}{2}}$)

$$\psi_{\frac{3}{2}} = \sum_{j=1,2}\left\{\phi_{\frac{3}{2}j}\frac{dA_j}{d\tau}E_j + \phi_{1j}A_{\frac{3}{2}j}E_j\right\} + \text{conjugates}, \qquad (3.3)$$

where the $A_{\frac{3}{2}j}(\tau)$ are arbitrary and the $\phi_{\frac{3}{2}j}$ satisfy

$$(\psi_0' - c)L_j\phi_{\frac{3}{2}j} = \frac{i}{\alpha_j}\left\{\frac{\psi_0'''}{\psi_0' - c} + \frac{2g\rho_0'}{\rho_0(\psi_0' - c)^2}\right\}\phi_{1j}. \qquad (3.4)$$

The orthogonality condition discussed in §2 demands that

$$M_j = \int_{-\infty}^{\infty}\left\{\frac{\psi_0'''}{(\psi_0' - c)^2} + \frac{2g\rho_0'}{\rho_0(\psi_0' - c)^3}\right\}\phi_{1j}^2\,dy = 0. \qquad (3.5)$$

This integral appears in the calculation of the growth rate of a linear disturbance with wave-number close to that of the neutral disturbance (cf. Howard 1963, equation (13) and thereafter), and we shall interpret any singularity in the customary manner. We then obtain

$$M_{\mathrm{I}} = -10\int_{-\infty}^{\infty} \tanh y \operatorname{sech}^4 y\,dy = 0, \tag{3.6}$$

$$M_{\mathrm{II}} = -10\int_{-\infty}^{\infty} \operatorname{cotanh} y \operatorname{sech}^6 y\,dy = -10\pi i. \tag{3.7}$$

The fact that $M_{\mathrm{II}} \neq 0$ demands that

$$A_{\mathrm{II}} = A_0, \text{a constant}. \tag{3.8}$$

In other words, no change of $O(\epsilon^{\frac{1}{2}})$ occurs in the amplitude of the mode II disturbance, as the solution (3.1) indicates. Because $M_{\mathrm{I}} = 0$, a solution for $\phi_{\frac{3}{2}\mathrm{I}}$ exists; it is, for $\beta g = \frac{4}{3}$,

$$\phi_{\frac{3}{2}\mathrm{I}} = -\tfrac{5}{2} i \operatorname{sech} y. \tag{3.9}$$

The vanishing of M_{I} led to the solution (3.1). If only a linear dependence upon time had been assumed, M_{I} would have multiplied the constant b, resulting in an indeterminate situation. The quadratic dependence eliminates this difficulty. It should be mentioned that Watson (1960, end of §2.1) has discussed the implications of a similar situation arising for the problem of self-interaction of slightly unstable disturbances. He concluded that velocity functions which become singular as $c_i \to 0$ should be included in the analysis. This behaviour in turn dictates that any equilibrium amplitude should be of $O(c_i)$. Here we are not concerned with the problem of third-order self-interaction and seek to investigate whether or not the second-order interaction of neutral disturbances can lead to instability. This leads us to consider the present solution, for which the rate of change of each disturbance is allowed to differ, at least initially.

Proceeding now to terms of $O(\epsilon^2)$, we express ψ_2 as (2.27), where $\phi_{2j} = \phi_{2j}(y,\tau)$. After solving for ρ_{2j} in terms of ϕ_{2j}, we obtain

$$\begin{aligned}(\psi_0'-c)L_{\mathrm{I}}\phi_{2\mathrm{I}} = \frac{i}{\alpha_{\mathrm{I}}}\left[\left\{\frac{\psi_0'''}{\psi_0'-c}+\frac{2g\rho_0'}{\rho_0(\psi_0'-c)^2}\right\}\phi_{\frac{3}{2}\mathrm{I}}+\frac{i}{\alpha_{\mathrm{I}}}\left\{\frac{\psi_0'''}{(\psi_0'-c)^2}+\frac{3g\rho_0'}{\rho_0(\psi_0'-c)^3}\right\}\phi_{1\mathrm{I}}\right] \\ \times\frac{d^2A_{\mathrm{I}}}{d\tau^2}+\frac{i}{\alpha_{\mathrm{I}}}\left[\frac{\psi_0'''}{\psi_0'-c}+\frac{2g\rho_0'}{\rho_0(\psi_0'-c)^2}\right]\phi_{1\mathrm{I}}\frac{dA_{\frac{3}{2}\mathrm{I}}}{d\tau}+F(y)\phi_{1\mathrm{I}}\phi_{1\mathrm{II}}A_0\tilde{A}_{\mathrm{I}}\end{aligned} \tag{3.10}$$

and

$$(\psi_0'-c)L_{\mathrm{II}}\phi_{2\mathrm{II}} = \frac{i}{\alpha_{\mathrm{II}}}\left[\frac{\psi_0'''}{\psi_0'-c}+\frac{2g\rho_0'}{\rho_0(\psi_0'-c)^2}\right]\phi_{1\mathrm{II}}\frac{dA_{\frac{3}{2}\mathrm{II}}}{d\tau}+\tfrac{1}{2}F(y)\phi_{1\mathrm{I}}^2A_{\mathrm{I}}^2. \tag{3.11}$$

Upon application of the orthogonality condition, the integral multipling the derivative of $A_{\frac{3}{2}\mathrm{I}}$ is simply M_{I}, which vanishes. Thus, $A_{\frac{3}{2}\mathrm{I}}$ remains arbitrary to this order. The other integrals do not vanish, however, and the following equations are obtained:

$$\frac{d^2A_{\mathrm{I}}}{d\tau^2}+\frac{16}{385}A_0\tilde{A}_{\mathrm{I}} = 0, \tag{3.12}$$

$$\frac{dA_{\frac{3}{2}\mathrm{II}}}{d\tau}-\frac{32}{525\pi}A_{\mathrm{I}}^2 = 0. \tag{3.13}$$

Upon consideration of the real and imaginary parts of (3.12), it is easily shown that one root of the characteristic equation indicates that A_{I} can grow exponentially at the rate $(16|A_0|/385)^{\frac{1}{2}}$. Although the interaction which excites the mode I disturbance is similar to that occurring in parametric resonance phenomena, the governing equation (3.12) is one order greater than that which typically governs such phenomena. This leads to the occurrence of oscillatory solutions, as well as the exponentially damped and growing solutions characteristic of this type of interaction.

The resonance which occurs in mode II is clearly of the classical type. For instance, if we set $A_0 = 0$ and take A_{I} to be constant, (3.13) predicts that the amplitude of the mode II disturbance will grow as $\epsilon^2 t$. On the other hand, when A_{I} grows exponentially, it can be shown that the most rapidly growing term in $A_{\frac{3}{2}\mathrm{II}}$ will have real and imaginary parts opposite in sign to $A_{0,r}$ and $A_{0,i}$, i.e. it will tend to reduce the magnitude of the mode II disturbance. Hence, the mode II disturbance acts only initially as a catalyst for the transfer of energy from the mean flow into mode I (which occurs because $\phi_{\frac{3}{2}\mathrm{I}}$ has the right form so as to give a non-zero Reynolds stress term in the energy transfer equation). While appreciable reduction of the amplitude of the mode II disturbance would weaken the transfer mechanism, such reduction would occur only after A_{I} had grown to a very large size relative to A_0.

The experimental results of Browand (1966) concerning subharmonic oscillations in a homogeneous free shear layer indicate that the amplitude of the primary disturbance remains relatively constant during the growth of the subharmonic (see his figure 14). Contrary to the above result, however, the primary decreases rapidly once the two disturbances are of comparable size.

For $\beta g = \frac{10}{3}$, it can be seen from figure 3 of Miles (1963) that three modes, with $\alpha_{\mathrm{I}} = 1$, $\alpha_{\mathrm{II}} = 2$, $\alpha_{\mathrm{III}} = 3$, can interact resonantly. The growth rate of mode II would now exceed those of the other two, and this fact leads to the conclusion that only algebraic growth is possible. For the more general case, however, when the growth rate of each disturbance is of the same order of magnitude, such a conclusion would not necessarily be valid.

4. Interaction of waves with fixed frequency

The previous analysis concerns a very atypical case, and a secular rate of change of $O(\epsilon)$ due to the interaction would seem to be more customary. This does occur if we consider how the interaction affects the streamwise behaviour of waves with fixed frequency. In order to strike the greatest contrast with the results of the previous section, we shall take the constant U_0 in (2.23) to be zero. In the appendix to a previous paper (Kelly 1967), it was argued, for the linear, homogeneous case, that only spatially decaying solutions are then possible and that these involve a change in frequency from the monotonically growing temporal solutions. Nevertheless, there are solutions to the non-linear problem which give rise to a slow spatial modulation of the neutral disturbances, as we shall now show.

If we now allow A_{I} and A_{II} in (2.10), (2.11) to become functions of the variable

$\hat{x} = \epsilon x$ and expand as in (2.6), (2.7), (2.27), the equations for $\phi_{2\mathrm{I}}$ and $\phi_{2\mathrm{II}}$, with $c = 0$, become

$$\psi_0' L_{\mathrm{I}} \phi_{2\mathrm{I}} = -2i\psi_0' \phi_{1\mathrm{I}} \frac{dA_{\mathrm{I}}}{d\hat{x}} + F(y)\, \phi_{1\mathrm{I}} \phi_{1\mathrm{II}} A_{\mathrm{II}} \tilde{A}_{\mathrm{I}} \tag{4.1}$$

and

$$\psi_0' L_{\mathrm{II}} \phi_{2\mathrm{II}} = -4i\psi_0' \phi_{1\mathrm{II}} \frac{dA_{\mathrm{II}}}{d\hat{x}} + \tfrac{1}{2} F(y)\, \phi_{1\mathrm{I}}^2 A_{\mathrm{I}}^2. \tag{4.2}$$

The orthogonality condition demands that

$$-2iQ_{\mathrm{I}} \frac{dA_{\mathrm{I}}}{d\hat{x}} + M A_{\mathrm{II}} \tilde{A}_{\mathrm{I}} = 0 \tag{4.3}$$

and

$$-8iQ_{\mathrm{II}} \frac{dA_{\mathrm{II}}}{d\hat{x}} + M A_{\mathrm{I}}^2 = 0, \tag{4.4}$$

where M is given by (2.33), (2.35) and

$$Q_j = \int_{-\infty}^{\infty} \phi_{1j}^2\, dy. \tag{4.5}$$

Thus, the Q_j are real, positive quantities for either mode. This fact gives rise to a conservation condition as follows: multiply (4.3) by $\tilde{A}_{\mathrm{I}}$ and use the conjugate of (4.4) to obtain

$$\tilde{A}_{\mathrm{I}} \frac{dA_{\mathrm{I}}}{d\hat{x}} + 4\left(\frac{\tilde{Q}_{\mathrm{II}}}{Q_{\mathrm{I}}}\right) A_{\mathrm{II}} \frac{d\tilde{A}_{\mathrm{II}}}{d\hat{x}} = 0. \tag{4.6}$$

If $\tilde{Q}_{\mathrm{II}} = \alpha Q_{\mathrm{I}}$, where α is real, as in our case, we obtain by adding the conjugate of (4.6) to (4.6) and integrating,

$$|A_{\mathrm{I}}|^2 + 4\alpha\, |A_{\mathrm{II}}|^2 = \Lambda_0, \tag{4.7}$$

where Λ_0 represents some initial value and, for our case, $\alpha = 2$.

This result allows us to make a conjecture concerning the more general case when the local Richardson number $J_l(y) > \frac{1}{4}$, the flow is stable on a linear basis, and only non-singular neutral modes can exist (Miles 1961). If secular rate of change of $O(\epsilon)$ had occurred for the temporal case, equations similar to (4.3), (4.4) would have resulted with the M_j (3.5) replacing the Q_j and $A_j = A_j(\hat{t})$, $\hat{t} = \epsilon t$. If non-singular modes interact, the M_j and M would be real, and a relation similar to (4.7) would hold. Thus, it appears unlikely that energy can be transferred from the mean flow due to resonant, second-order interaction if $J_l(y)$ exceeds $\frac{1}{4}$ by a term of $O(\epsilon)$.

The situation is quite different if we consider singular neutral modes, for which a condition like (4.7) would appear to be impossible unless $(\tilde{M}_{\mathrm{II}}/M_{\mathrm{I}})$ is real. One can show that this exceptional case does occur for Hølmboe's shear flow (Miles 1963), for which resonant interaction appears to be possible with $J_0 = \frac{2}{9}$, $\alpha_{\mathrm{I}} = \frac{1}{3}$, $\alpha_{\mathrm{II}} = \frac{2}{3}$. For that case, however, a more careful examination of the interaction integral is necessary, due to its singular behaviour. Also, as demonstrated by the concurrent work of A. Craik, this behaviour can only be aggravated if the interaction of three-dimensional disturbances is considered.

The solution of (4.3), (4.4) might be of interest as being representative of the transfer of energy between disturbances for the non-singular case. If we concentrate on obtaining a solution for $|A_{\mathrm{I}}|^2$, make use of (4.3), its conjugate, and (4.7) in an identity for $(d^2/d\hat{x}^2)(|A_{\mathrm{I}}|)^2$, we can derive the equation

$$\frac{d^2}{d\hat{x}^2}|A_{\mathrm{I}}|^2 - \frac{64}{1225}\{2\Lambda_0 - 3|A_{\mathrm{I}}|^2\}|A_{\mathrm{I}}|^2 = 0. \tag{4.8}$$

Although this equation is satisfied by the Weierstrassian elliptic function, it is sufficient for our purposes to consider its singular points. Using the customary phase plane terminology (Stoker 1950, chapter III*b*), we can say that the origin is a saddle point, a centre exists at $|A_{\mathrm{I}}|^2 = 2\Lambda_0/3$, and that closed trajectories exist in the phase plane for $0 < |A_{\mathrm{I}}|^2 < \Lambda_0$. The fact that a centre exists indicates that strong amplitude modulation is more likely for cases when the relative amplitude of one disturbance is initially small.

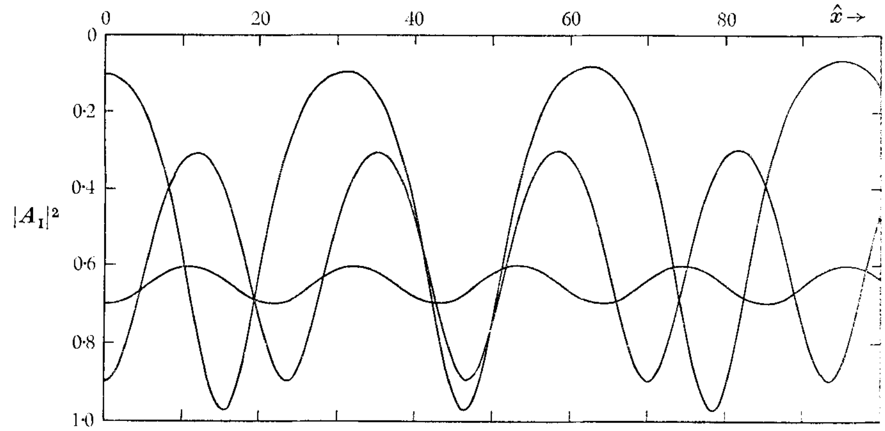

FIGURE 1

Solutions to (4.8) were obtained on an E.A.I. PACE TR-48 electronic analog at the U.S. Naval Electronics Laboratory, San Diego, for $\Lambda_0 = 1$. The initial conditions imposed were that the initial gradient of $|A_{\mathrm{I}}|^2$ was zero and that $|A_{\mathrm{I}}|^2$ had, in turn, the values 0·1, 0·7, 0·9 at $\hat{x} = 0$. The solutions are shown in figure 1. The curve for the intermediate value indicates very slight amplitude modulation.

5. Conclusions

Our conclusions are necessarily limited by the special nature of the stratified shear flow which has been considered. None the less, it does appear that resonant wave interaction, when a mean flow is present, is more pertinent to the growth of unstable disturbances than to the destabilization of, say, gravity waves. It also appears likely, on the basis of the solution obtained in § 3 and the discussion of § 4, that additional energy transfer from the free stream to reasonantly interacting unstable disturbances can occur. Further, for the case considered, the effects of such transfer can apparently be felt at a lower order than that at which the interaction occurs. The example of a homogeneous jet indicates, however, that satisfaction of the conditions regarding the eigenvalues need not necessarily imply that such transfer occurs.

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